Saginaw Valley State University
2016 Math Olympics — Level II Solutions

1. Find the value of the product $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{n^2}\right)$.

   (a) $\frac{n-1}{n^2}$ **(b)** $\frac{n+1}{n^2}$ **(c)** $\frac{n+1}{2n}$ **(d)** $\frac{n-1}{2n}$ **(e)** None of the above

   SOLUTION **(c)**: Using the identity $a^2 - b^2 = (a-b)(a+b)$ in each factor, we have

$$\begin{aligned}\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{n^2}\right) &= \left(\frac{1}{2}\cdot\frac{3}{2}\right)\left(\frac{2}{3}\cdot\frac{4}{3}\right)\left(\frac{3}{4}\cdot\frac{5}{4}\right)\cdots\left(\frac{n-1}{n}\cdot\frac{n+1}{n}\right)\\ &= \frac{1}{2}\cdot\frac{n+1}{n} = \frac{n+1}{2n}\end{aligned}$$

   after canceling all intermediate factors.

2. Let x and y be real numbers such that $x+y=3$, and $x^3+y^3=117$. What is the value of x^2+y^2?

   (a) 29 **(b)** 3 **(c)** 9 **(d)** 17 **(e)** 45

   SOLUTION **(a)**: Factor the second equation and use the first equation. We have $x^3+y^3=(x+y)(x^2-xy+y^2)=3(x^2-xy+y^2)=117$. Therefore $39=x^2-xy+y^2=(x+y)^2-3xy=9-3xy$. Thus, $xy=-10$. Substituting this into $x^2-xy+y^2=39$ we find that $x^2+y^2=29$.

3. What is the exact value of $\dfrac{\sin(3\alpha)+\sin(\alpha)}{\sin(2\alpha)\cos(\alpha)}$ (for all values of α such that the above expression is defined)?

   (a) -1 **(b)** 0 **(c)** 1 **(d)** 2 **(e)** None of the above

   SOLUTION **(d)**:

$$\begin{aligned}\frac{\sin(3\alpha)+\sin(\alpha)}{\sin(2\alpha)\cos(\alpha)} &= \frac{\sin(2\alpha+\alpha)+\sin(2\alpha-\alpha)}{\sin(2\alpha)\cos(\alpha)}\\ &= \frac{\sin(2\alpha)\cos(\alpha)+\cos(2\alpha)\sin(\alpha)+\sin(2\alpha)\cos(\alpha)-\cos(2\alpha)\sin(\alpha)}{\sin(2\alpha)\cos(\alpha)}\\ &= \frac{2\sin(2\alpha)\cos(\alpha)}{\sin(2\alpha)\cos(\alpha)} = 2.\end{aligned}$$

4. Three fair dice are rolled. What is the probability that the product of the three outcomes is a prime number? Recall that 1 is not considered to be prime.

   (a) 0 **(b)** $\frac{1}{72}$ **(c)** $\frac{1}{36}$ **(d)** $\frac{1}{24}$ **(e)** None of the above

   SOLUTION **(d)**: There are $6^3 = 216$ ordered triples of dice rolls. The product is prime precisely when two rolls are 1 and the third is a prime number. Since the prime 2,3, or 5 can appear in any of the three positions, there are 9 such triples, so the probability is $\frac{9}{216} = \frac{1}{24}$.

5. What is the sum of the digits of all numbers from 1 to 1000?

(a) 13501 **(b)** 13601 **(c)** 13701 **(d)** 13801 **(e)** None of the above

SOLUTION **(a)**: Let us add zero to the set of the numbers and divide the set into 500 pairs $(0, 999)$, $(1, 998)$,…, $(499, 500)$. The sum of the digits in each pair is 27. In addition, we need to take into account 1000, so that the total sum of all digits will be $27 \cdot 500 + 1 = 13501$.

6. In order that $1651817N833536$ equal 2016^4, the letter N should be replaced by the digit

(a) 3 **(b)** 4 **(c)** 6 **(d)** 7 **(e)** 8

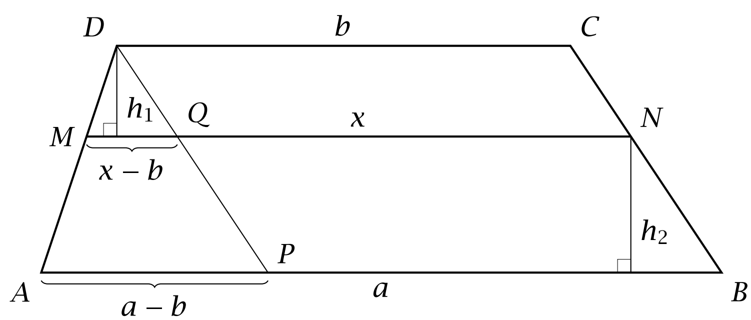
SOLUTION **(c)**: Since 2016 is divisible by 3, all its higher powers are divisible by 9. Noting that any positive integer differs from the sum of its digits by a multiple of 9, it follows that the sum of the digits of 2016^4 must also be divisible by 9. This requires that N be replaced by 6.

7. A trapezoid has bases a and b. Which of the following gives the length of the line segment parallel to both a and b that splits the area of the trapezoid in two pieces of ratio $2:3$?

(a) $a + \frac{2}{3}(b-a)$ **(b)** $\frac{2a^2+3b^2}{6}$ **(c)** $\sqrt{\frac{2a^2+3b^2}{5}}$ **(d)** $\frac{2a+3b}{5}$

(e) $\sqrt{\frac{2a+3b}{6}}$

SOLUTION **(c)**: We are looking for the length of the segment MN in the following figure, where the ratio of the areas of $NCDM$ and $ABNM$ is $2:3$.



$$\frac{2}{3} = \frac{A_{MNCD}}{A_{ABNM}} = \frac{\frac{x+b}{2} \cdot h_1}{\frac{a+x}{2} \cdot h_2} = \frac{x+b}{x+a} \cdot \frac{h_1}{h_2}$$

or

$$\frac{x+a}{x+b} \cdot \frac{h_2}{h_1} = \frac{3}{2}. \tag{1}$$

Since $MQ \parallel AP$, the triangles $\triangle MQD$ and $\triangle APD$ are similar, which means

$$\frac{a-b}{x-b} = \frac{h_1+h_2}{h_1} = 1 + \frac{h_2}{h_1}$$

so

$$\frac{h_2}{h_1} = \frac{a-b}{x-b} - 1 = \frac{a-b-x+b}{x-b} = \frac{a-x}{x-b}. \tag{2}$$

Combining (1) and (2) gives us

$$\frac{x+a}{x+b} \cdot \frac{a-x}{x-b} = \frac{a^2-x^2}{x^2-b^2} = \frac{3}{2}.$$

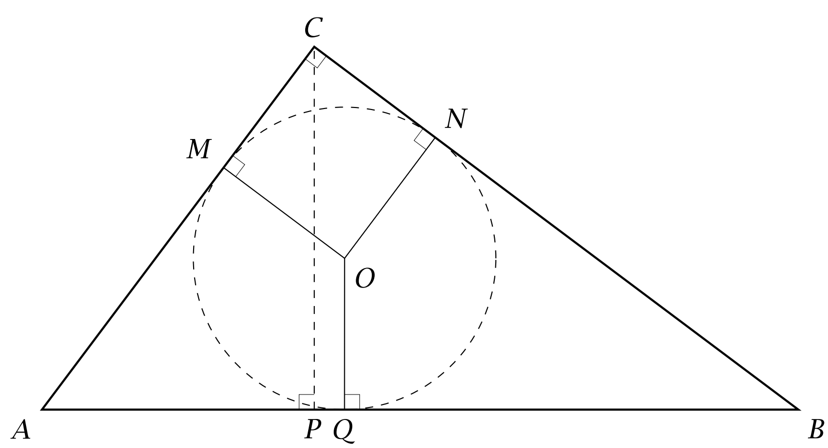
Then

$$\begin{aligned} 2(a^2-x^2) &= 3(x^2-b^2) \\ 2a^2-2x^2 &= 3x^2-3b^2 \\ 2a^2+3b^2 &= 3x^2+2x^2 \\ 5x^2 &= 2a^2+3b^2 \\ x &= \sqrt{\frac{2a^2+3b^2}{5}} \end{aligned}$$

8. The lengths of the legs of a right triangle are 15 and 20, respectively. What is the distance from the center of the circle inscribed in the triangle to the altitude of the triangle towards the hypotenuse?

(a) 0 **(b)** $\sqrt{2}$ **(c)** 2 **(d)** $2\sqrt{3}$ **(e)** None of the above

SOLUTION **(e)**:



The distance between the incenter O and the altitude CP is the same as the distance between the points P and Q. To find PQ, we can find AP and AQ and subtract them.
We know that $AQ = AM$, $MC = CN$ and $BN = QB$, therefore

$$\begin{aligned} AQ + QB \phantom{{}+MC} &= 25 \\ AQ \phantom{{}+QB} + MC &= 15 \\ QB + MC &= 20 \end{aligned}$$

Solving this system gives us $AQ = 10$, $QB = 15$ and $MC = 5$.
The area of the triangle $\triangle ABC$ is $\frac{1}{2} \cdot AB \cdot CP$, and also $\frac{1}{2} \cdot AC \cdot BC$, so

$$25 \cdot CP = 15 \cdot 20$$
$$CP = 12$$

According to the Pythagorean theorem,

$$\begin{aligned} AP^2 + CP^2 &= AC^2 \\ AP^2 + 144 &= 225 \\ AP^2 &= 225 - 144 = 81 \\ AP &= 9 \end{aligned}$$

Finally, $PQ = AQ - AP = 10 - 9 = 1$.

9. Simplify $\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} - \frac{1}{2}\cos\alpha}}$ for $0 \le \alpha \le 2\pi$.

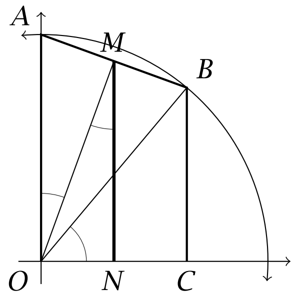
(a) $\cos\alpha$ **(b)** $\sin\alpha$ **(c)** $\cos\left(\frac{\pi}{4} - \frac{\alpha}{4}\right)$

(d) $\sin\left(\frac{\alpha}{4}\right)$ **(e)** None of the above

SOLUTION **(c)**:

$$\begin{aligned} \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} - \frac{1}{2}\cos\alpha}} &= \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1 - \cos\alpha}{2}}} \\ &= \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\sin^2\frac{\alpha}{2}}} \\ &= \sqrt{\frac{1}{2} + \frac{1}{2}\sin\frac{\alpha}{2}} \quad \text{since } 0 \le \tfrac{\alpha}{2} \le \pi \text{ and so } \sin\tfrac{\alpha}{2} \ge 0 \\ &= \sqrt{\frac{1 + \sin\frac{\alpha}{2}}{2}} \end{aligned}$$

The quantity $\frac{1+\sin\frac{\alpha}{2}}{2}$ is the arithmetic average of 1 and $\sin\frac{\alpha}{2}$.



In the picture on the left, $A = (0, 1)$ and the angle $\angle BOC = \frac{\alpha}{2}$, so

$$OA = 1$$
$$BC = \sin\frac{\alpha}{2}$$

The point M is the midpoint of AB so

$$MN = \frac{1 + \sin\frac{\alpha}{2}}{2}$$

The angles $\angle AOM$ and $\angle OMN$ are both equal to $\frac{1}{2}\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) = \frac{\pi}{4} - \frac{\alpha}{4}$.

Then

$$\begin{aligned} OM &= OA \cdot \cos\left(\frac{\pi}{4} - \frac{\alpha}{4}\right) \\ &= 1 \cdot \cos\left(\frac{\pi}{4} - \frac{\alpha}{4}\right) \\ &= \cos\left(\frac{\pi}{4} - \frac{\alpha}{4}\right) \end{aligned}$$

and

$$\begin{aligned} MN &= OM \cdot \cos\left(\frac{\pi}{4} - \frac{\alpha}{4}\right) \\ &= \cos\left(\frac{\pi}{4} - \frac{\alpha}{4}\right) \cdot \cos\left(\frac{\pi}{4} - \frac{\alpha}{4}\right) \\ &= \cos^2\left(\frac{\pi}{4} - \frac{\alpha}{4}\right) \end{aligned}$$

Since $-\frac{\pi}{2} \leq \frac{\pi}{4} - \frac{\alpha}{4} \leq \frac{\pi}{2}$,

$$\sqrt{\cos^2\left(\frac{\pi}{4} - \frac{\alpha}{4}\right)} = \cos\left(\frac{\pi}{4} - \frac{\alpha}{4}\right)$$

10. $\ln(\tan 1^\circ) + \ln(\tan 2^\circ) + \cdots + \ln(\tan 88^\circ) + \ln(\tan 89^\circ) =$

   (a) 0 **(b)** 1 **(c)** $\ln 2$ **(d)** $\frac{\sqrt{2}}{2}$ **(e)** None of the above

   SOLUTION **(a)**:

$$
\begin{aligned}
&\ln(\tan 1^\circ) + \ln(\tan 2^\circ) + \cdots + \ln(\tan 88^\circ) + \ln(\tan 89^\circ) \\
&= \ln\left((\tan 1^\circ)\cdot(\tan 2^\circ)\cdot\cdots\cdot(\tan 88^\circ)\cdot(\tan 89^\circ)\right) && \text{since } \ln(a)+\ln(b)=\ln(ab) \\
&= \ln\left(\frac{\sin 1^\circ}{\cos 1^\circ}\cdot\frac{\sin 2^\circ}{\cos 2^\circ}\cdot\cdots\cdot\frac{\sin 88^\circ}{\cos 88^\circ}\cdot\frac{\sin 89^\circ}{\cos 89^\circ}\right) && \text{since } \tan\alpha = \tfrac{\sin\alpha}{\cos\alpha} \\
&= \ln\left(\frac{\sin 1^\circ}{\sin 89^\circ}\cdot\frac{\sin 2^\circ}{\sin 88^\circ}\cdot\cdots\cdot\frac{\sin 88^\circ}{\sin 2^\circ}\cdot\frac{\sin 89^\circ}{\sin 1^\circ}\right) && \text{since } \cos\alpha = \sin(90^\circ-\alpha) \\
&= \ln\left(\frac{(\sin 1^\circ)(\sin 2^\circ)\cdots(\sin 88^\circ)(\sin 89^\circ)}{(\sin 89^\circ)(\sin 88^\circ)\cdots(\sin 2^\circ)(\sin 1^\circ)}\right) \\
&= \ln(1) = 0
\end{aligned}
$$

11. Find the sum of all positive 2 digit integers that are divisible by each of their digits.

   (a) 495 **(b)** 546 **(c)** 582 **(d)** 630 **(e)** None of the above

   SOLUTION **(d)**: Let n be a positive 2 digit integer with decimal representation AB such that $A \div n$ and $B \div n$. Since n has decimal representation AB, $n = 10A + B$. None of A, B can be 0.
   Since A n and obviously A $10A$, A must divide B.
   Since B n and obviously B B, B must divide $10A$.
   For each of the 9 possible values of A we need to find all the values of B that are multiple of A and a factor $10A$ at the same time.

A	**Multiples of A**	**Factors of $10A$**	**Possible B's**
1	any	1, 2, 5	1, 2, 5
2	2, 4, 6, 8	1, 2, 4, 5	2, 4
3	3, 6, 9	1, 2, 3, 5, 6	3, 6
4	4, 8	1, 2, 4, 5, 8	4, 8
5	5	1, 2, 5	5
6	6	1, 2, 3, 4, 5, 6	6
7	7	1, 2, 5, 7	7
8	8	1, 2, 4, 5, 8	8
9	9	1, 2, 3, 5, 6, 9	9

   All the possible values of n are 11, 12, 15, 22, 24, 33, 36, 44, 48, 55, 66, 77, 88, and 99. The sum of these numbers is

$$(11 + 22 + \cdots + 99) + (12 + 24 + 36 + 48) + 15 = 11 \cdot 45 + 12 \cdot 10 + 15 = 495 + 120 + 15 = 630$$

12. Which of the following is not equal to $\sin^{-1}\left(\sin\frac{46\pi}{7}\right)$?

(a) $\frac{46\pi}{7}$ **(b)** $\frac{3\pi}{7}$

(c) $-\tan^{-1}\left(\tan\frac{4\pi}{7}\right)$ **(d)** $\cos^{-1}\left(\sqrt{1-\sin^2\left(\frac{4\pi}{7}\right)}\right)$

(e) They are all equal to $\sin^{-1}\left(\sin\frac{46\pi}{7}\right)$

SOLUTION **(a)**: Clearly $\frac{46\pi}{7}$ is not in the range of $\sin^{-1}$ and so **(a)** cannot be correct.
To see that the others are equivalent:

- $\frac{46\pi}{7} = 7\pi - \frac{3\pi}{7}$, so $\sin\left(\frac{46\pi}{7}\right) = \sin\left(\frac{3\pi}{7}\right)$ and $\frac{3\pi}{7}$ is between 0 and $\frac{\pi}{2}$.
- $\frac{4\pi}{7} = \pi - \frac{3\pi}{7}$, $\frac{4\pi}{7}$ has the same terminal side as $\frac{46\pi}{7}$. However $\tan\left(\frac{4\pi}{7}\right)$ is negative because $\frac{4\pi}{7}$ is in the second quadrant. So $\tan^{-1}\left(\tan\frac{4\pi}{7}\right) = -\frac{3\pi}{7}$. So $\frac{3\pi}{7} = -\tan^{-1}\left(\tan\frac{4\pi}{7}\right)$.
- If $\theta = \frac{3\pi}{7}$, $\sin\theta = sin\frac{3\pi}{7}$ and we have $\sin^2\theta + \cos^2\theta = 1$, or $\sin^2\frac{3\pi}{7} + \cos^2\theta = 1$. Solving for θ and using the fact that $\frac{3\pi}{7}$ is in the right range for $\cos^{-1}\theta$ gives

$$\theta = \cos^{-1}\left(\sqrt{1-\sin^2\left(\frac{4\pi}{7}\right)}\right).$$

13. In the expansion of $\left(x^3 + \frac{y}{x^2}\right)^{15}$, what is the exponent on y in the term that has no factors of x?

(a) 5 **(b)** 6 **(c)** 9 **(d)** 10 **(e)** None of the above

SOLUTION **(c)**:

$$\begin{aligned}\left(x^3 + \frac{y}{x^2}\right)^{15} &= \sum_{k=0}^{15}\binom{15}{k}\left(x^3\right)^{15-k}\left(\frac{y}{x^2}\right)^k \\ &= \sum_{k=0}^{15}\binom{15}{k}x^{45-3k}y^k x^{-2k}\end{aligned}$$

So on each term, when the exponent on y is k, the exponent on x is $45-3k+(-2k)$. We need to solve for k that results in $45-3k-2k=0$ to find which k gives no factors of x. This gives us $45-5k=0$, so $k=9$.

14. When we were preparing this test, we lost the question that went with these answers. Fortunately, since there must be exactly one correct answer for any question, you can decide which one answer below is true.

    (a) The cow jumped over the moon.

    (b) The cow jumped over the moon, or the little dog laughed to see such a sight.

    (c) The little dog laughed to see such a sight, or the dish ran away with the spoon.

    (d) The dish ran away with the spoon and the cow jumped over the moon.

    (e) If the cow didn't jump over the moon, then the little dog laughed to see such a sight.

    SOLUTION **(c)**: We know that exactly one of the above statements is true. It cannot be **(a)**, since if **(a)** is true, **(b)** and **(e)** will be true as well, and only one of the statements can be true.
    If **(b)** was true, then **(a)** or **(c)** would be true as well. Again, that is not possible, since only one of the statements can be true.
    If **(d)** was true, then **(a)** would be true as well, which cannot happen.
    If **(e)** was true, then, since we already know that **(a)** is false, "the little dog laughed to see such a sight" would have to be true, and **(c)** would be true as well. So **(e)** cannot be true.
    We ruled out **(a)**, **(b)**, **(d)** and **(e)**. Since one of them must be true, it has to be **(c)**.

15. $\left(\log_{\frac{1}{5}} 9\right) \cdot \left(\log_{27} \frac{1}{5}\right) =$

    (a) $\frac{1}{2}$

    (b) $\frac{2}{3}$

    (c) 3

    (d) Since $\frac{1}{5}$ cannot be the base of a logarithm, this question has no answer.

    (e) None of the above

    SOLUTION **(b)**: According to the rules of logarithms,

$$\left(\log_{\frac{1}{5}} 9\right) \cdot \left(\log_{27} \frac{1}{5}\right) = \log_{27} \left(\frac{1}{5}\right)^{\log_{\frac{1}{5}} 9} = \log_{27} 9 = \frac{2}{3}$$

16. If $\sin^2\theta = \frac{1}{3}$, what is $72\cos(2\theta)$?

(a) 0 **(b)** 24 **(c)** 72 **(d)** 124 **(e)** None of the above

SOLUTION **(b)**:

$$72\cos(2\theta) = 72\left(1 - 2\sin^2\theta\right) = 72\left(1 - \frac{2}{3}\right) = \frac{72}{3} = 24$$

17. Find the number of solutions of the equation

$$\log_{\sqrt{2}\sin x}(1 + \cos x) = 2$$

on the interval $[0, 2\pi)$.

(a) 0 **(b)** 1 **(c)** 2 **(d)** 3 **(e)** None of the above

SOLUTION **(b)**: The equation is equivalent to the equation

$$\left(\sqrt{2}\sin x\right)^2 = 1 + \cos x$$

if and only if $\sqrt{2}\sin x > 0$. To get all the solution of the original equation on $[0, 2\pi)$, we will find all solutions of the equation

$$\left(\sqrt{2}\sin x\right)^2 = 1 + \cos x$$

on $(0, \pi)$:

$$\begin{aligned}
\left(\sqrt{2}\sin x\right)^2 &= 1 + \cos x \\
2\sin^2 x &= 1 + \cos x \\
2(1 - \cos^2 x) &= 1 + \cos x \\
2\cos^2 x + \cos x - 1 &= 0 \\
(2\cos x - 1)(\cos x + 1) &= 0 \\
\cos x &= \frac{1}{2}, -1
\end{aligned}$$

The only solution of this equation on $(0, \pi)$ is $x = \frac{\pi}{3}$.

18. Consider the four functions:

$$f(x) = \log_7 x$$
$$g(x) = \log_7(-x)$$
$$h(x) = \log_{\frac{1}{7}} x$$
$$k(x) = \log_{\frac{1}{7}}(-x)$$
$$l(x) = -\log_{\frac{1}{7}} x$$

Which of these functions are increasing on their domain?

(a) $f(x)$ only **(b)** $f(x)$ and $h(x)$ only **(c)** $f(x)$ and $k(x)$ only

(d) $f(x)$ and $l(x)$ only **(e)** $f(x)$, $k(x)$ and $l(x)$ only

SOLUTION **(e)**: The function $\log_a x$ is increasing if and only if $a > 1$. That means $f(x)$ is increasing and $h(x)$ is decreasing.
If a function $F(x)$ increases, both $F(-x)$ and $-F(x)$ decreases. Similarly, if a function $F(x)$ decreases, both $F(-x)$ and $-F(x)$ increases.
So $g(x)$ is decreasing, and $k(x)$ and $l(x)$ are both increasing.

19. Let $N = 2^2 3^3 7^4$. How many factors of N^2 are less than N but not factors of N?

(a) 72 **(b)** 75 **(c)** 97 **(d)** 98 **(e)** None of the above

SOLUTION **(d)**: Any factor of N will be of the form $2^a 3^b 7^c$ where $0 \le a \le 2$, $0 \le b \le 3$ and $0 \le c \le 4$, so there are $3 \cdot 4 \cdot 5 = 60$ factors. Out of those, 59 are less than N.
Any factor of N^2 will be of the form $2^k 3^m 7^n$ where $0 \le k \le 4$, $0 \le m \le 6$ and $0 \le n \le 8$, so there are $5 \cdot 7 \cdot 9 = 315$ factors of N^2. For each factor A of N^2 that is less than N, there is a corresponding factor B of N^2 that is larger than N, so that $AB = N^2$. Out of the 315 factors of N^2, there are 157 factors less than N, 1 factor equal to N, and 157 factors larger than N.
Since every factor of N is also a factor of N^2, there are $157 - 59 = 98$ factors of N^2 that are less than N but are not factors of N.

20. Give the exact value of $\sin \frac{\pi}{10}$.

(a) $\dfrac{\sqrt{10 + 2\sqrt{5}}}{4}$ **(b)** $\dfrac{(\sqrt{5} - 1)\sqrt{10 + 2\sqrt{5}}}{8}$ **(c)** $\dfrac{\sqrt{5} - 1}{4}$

(d) $\dfrac{\sqrt{5} + 1}{4}$ **(e)** None of the above

SOLUTION **(c)**:

$$\begin{aligned}
\sin\frac{\pi}{10} &= \sin\left(\frac{5\pi}{10}-\frac{4\pi}{10}\right) = \sin\left(\frac{\pi}{2}-\frac{4\pi}{10}\right)\\
&= \sin\frac{\pi}{2}\cos\frac{4\pi}{10} - \cos\frac{\pi}{2}\sin\frac{4\pi}{10}\\
&= \cos\frac{4\pi}{10} = \cos\left(2\cdot\frac{2\pi}{10}\right)\\
&= 2\cos^2\frac{2\pi}{10} - 1\\
&= 2\left(1-2\sin^2\frac{\pi}{10}\right)^2 - 1
\end{aligned}$$

Let $x = \sin\frac{\pi}{10}$. Then

$$\begin{aligned}
x &= 2(1-2x^2)^2 - 1\\
x &= 2(1-4x^2+4x^4) - 1\\
x &= 1 - 8x^2 + 8x^4
\end{aligned}$$

or, after subtracting x and switching sides,

$$\begin{aligned}
8x^4 - 8x^2 - x + 1 &= 0\\
8x^2(x^2-1) - (x-1) &= 0\\
8x^2(x+1)(x-1) - (x-1) &= 0\\
(8x^3+8x^2-1)(x-1) &= 0\\
(8x^3+4x^2+4x^2-1)(x-1) &= 0\\
(4x^2(2x+1)+(2x+1)(2x-1))(x-1) &= 0\\
(4x^2+2x-1)(2x+1)(x-1) &= 0
\end{aligned}$$

The solutions of this equation are $x = 1$, $x = -\frac{1}{2}$ and $x = \frac{-1\pm\sqrt{5}}{4}$. Out of the four solutions, only $\frac{-1+\sqrt{5}}{4}$ can be equal to $\sin\frac{\pi}{10}$.

21. An infinite string of digits is formed by writing odd numbers in a row in the following way: you start with one 1, followed by three 3's, then five 5's, seven 7's, then nine 9's, eleven 11's *etc.* The beginning of the string is 13335555577.... What is the 2016th digit in the string?

(a) 3 **(b)** 4 **(c)** 5 **(d)** 6 **(e)** None of the above

Solution **(d)**: With single digit numbers, you create the string

1333555557777777999999999

which has $1+3+5+7+9 = 25$ digits.
There are 45 odd two-digit numbers, starting with 11 and ending with 99. The part of the string created with two digit numbers has

$$2(11 + 13 + \cdots + 97 + 99) = 2 \cdot 110 \cdot \frac{45}{2} = 4950 \text{ digits}$$

So using all the single and double digit odd numbers, we get a string with 4975 digits, which is more than 2016. So the 2016th digit will be somewhere in the part of the string formed by two-digit odd numbers.
If k is a two-digit odd number, then the length of the string formed using all numbers up to k and including the k copies of k is

$$\begin{aligned} 1 + 3 + 5 + 7 + 9 + 2(11 + 13 + \cdots + k) &= 2(1 + 3 + \cdots + k) - 25 \\ &= 2 \cdot (1 + k) \cdot \frac{1 + k}{4} - 25 \\ &= \frac{(1 + k)^2}{2} - 25 \end{aligned}$$

Solving the equation

$$\frac{(1 + k)^2}{2} - 25 = 2016$$

gives us

$$k = \sqrt{4082} - 1$$

Since $4096 = 64^2$, k will be somewhere between 62 and 63, which means somewhere in the part of the string created by 63 copies of 63, so the 2016th digit will be either 6 or 3.
Since the single digit numbers form the first 25 digits of the string, in the double digit number part of the string, the even position digits are formed by the tens digits of the numbers, and the odd position digits are the ones digits of the numbers. So the 2016th digit must be 6.

22. The sum of the solutions of the equation:

$$2\,(2^x) + 16\,(2^{-x}) = 33$$

is

(a) 3 **(b)** $-\frac{1}{3}\log_2(33)$ **(c)** 9

(d) There are no solutions **(e)** None of the above

SOLUTION **(a)**: Multiplying both sides of the equation by 2^x gives $2\,(2^x)^2 + 16 = 33\,(2^x)$
Let $u = 2^x$ and subtract $33u$ from both sides to get the quadratic equation:

$$2u^2 - 33u + 16 = 0$$

This may be factored as $(2u - 1)(u - 16) = 0$ giving the two solutions $u = \frac{1}{2}$ and $u = 16$.
Replacing u with 2^x gives us the two solutions $x = -1$ and $x = 4$.

23. A number X is chosen at random from the first 100 terms of the arithmetic sequence $2, 5, 8, \ldots$, and another number Y is chosen at random from the first 100 terms of the arithmetic sequence $3, 7, 11, \ldots$. Find $P(X = Y)$.

(a) .0021 **(b)** .0023 **(c)** .0025 **(d)** .0030 **(e)** .0033

SOLUTION **(c)**: The two sequences are

$$a_m = 2 + 3(m-1), m = 1, 2, \ldots, 100$$
$$b_n = 3 + 4(n-1), n = 1, 2, \ldots, 100$$

The common terms will form the sequence

$$c_k = 11 + 12(k-1), k = 1, 2, \ldots$$

The largest common term must be less than or equal to the smaller of a_{100} and b_{100}, which is $2 + 3 \cdot 99 = 299$.

$$\begin{aligned} 11 + 12(k-1) &\le 299 \\ 12(k-1) &\le 288 \\ k - 1 &\le \frac{288}{12} \\ k &\le 25 \end{aligned}$$

So out of $100 \times 100 = 10,000$ different pairs of values for X and Y, there are 25 cases where $X = Y$, so

$$P(X = Y) = \frac{25}{10000} = .0025$$

24. A writes to B and does not receive an answer. Assuming that one of every n letters sent is lost in the mail, find the probability that B received the letter. Assume that B would have answered the letter if he had received it.

(a) $\frac{n}{n-1}$ **(b)** $\frac{n-1}{n^2} + \frac{1}{n}$ **(c)** $\frac{n-1}{2n-1}$ **(d)** $\frac{n}{2n-1}$ **(e)** $\frac{n-1}{n^2}$

SOLUTION **(c)**: Probability of any individual letter being lost in the mail is $1/n$.
We are trying to find the conditional probability

$$P\left(B \text{ received the letter} | \text{no reply}\right)$$

The Bayes Theorem states that

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

Applying this gives us

$$P\left(B \text{ received the letter}|\text{no reply}\right) = \frac{P\left(\text{no reply}|B \text{ received the letter}\right) P\left(B \text{ received the letter}\right)}{P\left(\text{no reply}\right)}$$

All the probabilities on the right side are easy to calculate:

- The probability that there was no reply given that B received the letter is the probability of the B's reply being lost in the mail, which is $1/n$.
- The probability of B receiving the letter is the probability of the A's letter not being lost, which is $1 - \frac{1}{n} = \frac{n-1}{n}$.
- Finally

$$\begin{aligned} P\left(\text{no reply}\right) &= P\left(A\text{'s letter was lost}\right) + \\ &\quad P\left(A\text{'s letter was delivered}\right) P\left(B\text{'s letter was lost}|A\text{'s letter was delivered}\right) \\ &= \frac{1}{n} + \frac{n-1}{n}\frac{1}{n} \\ &= \frac{n}{n^2} + \frac{n-1}{n^2} \\ &= \frac{2n-1}{n^2} \end{aligned}$$

Substituting these three probabilities to the right side of the Bayes Theorem application, we get

$$P\left(B \text{ received the letter}|\text{no reply}\right) = \frac{\frac{1}{n}\frac{n-1}{n}}{\frac{2n-1}{n^2}} = \frac{\frac{n-1}{n^2}}{\frac{2n-1}{n^2}} = \frac{n-1}{2n-1}$$

25. In a survey of customer satisfactions, participants are asked to give a score of 1, 2, 3, or 4 to each of 6 questions. If participants are instructed not to give the same numerical score to more than 4 questions, how many responses are possible?

(a) 2304 **(b)** 3840 **(c)** 4020 **(d)** 4095 **(e)** 4096

Solution **(c)**: Without the restriction, there are 4 ways to answer each of the 6 questions, so there are 4^6 unrestricted responses.
There are two ways to violate the instructions: a person can give all 6 questions the same response, or the person can give 5 of the question the same response and one question different response.

- There are 4 ways to give all 6 questions the same response.

- To give 5 questions the same response and one question a different response, one has to first pick which of the question will receive a different response. There are 6 ways to pick. For each of those 6 ways, there are 4 possible answers for the selected question, and for each of the 4 answers, there are 3 possible answers to assign the remaining five questions. Altogether there are $6 \times 4 \times 3$ ways to give the same answer to exactly 5 questions.

There will be $4^6 - (6 \times 4 \times 3 + 4)$ valid responses. To simplify this, one can factor out 4 to get

$$4(4^5 - 6 \times 3 - 1) = 4(2^{10} - 19) = 4(1024 - 19) = 4 \times 1005 = 4020$$