# Fractals


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### Introduction

For much of the past two-and-one-half millennia, humans have viewed our geometrical world through the lens of a Euclidian paradigm. Despite knowing that our planet is spherically shaped, mathematicians have developed the mathematics of non-Euclidian geometry only within the past 200 years. Similarly, only in the past century have mathematicians developed an understanding of such common phenomena as snowflakes, clouds, coastlines, lightning bolts, rivers, patterns in vegetation, and the trajectory of molecules in Brownian motion (Peitgen 75). This branch of mathematics is known as fractal geometry. Fractals force us to alter our view of dimensionality, produce chaotic patterns from nearly identical starting positions, and have real world applications in nature and human creations.

A fractal is a geometrical shape that can be divided into parts, each of which is a smaller duplicate of the whole (Mandelbrot 15). This property of duplication is called self-similarity. In mathematics, a fractal is based upon an equation that undergoes an iteration or recursion – that is, a repetition of itself. The object does not have to exhibit exactly the same structure on all scales, but the same type of structures must be apparent on all scales for the structure to be considered a fractal (Connors). Fractals share certain features, including a fine structure at arbitrarily small scales, an irregular form not easily described in Euclidean geometric language, self-similarity, and recursive properties (Falconer 12). A fractal reveals greater complexity as it is enlarged. While Euclidean shapes resemble straight lines the more closely we examine them, fractals expose greater details upon closer examination (Hopkins 23). Because they appear similar regardless of the level of magnification, fractals are considered to be infinitely complex (Chubb).

Back in the 17th century, mathematicians became fascinated by the mathematics that eventually led to fractals. Gottfried Leibniz wrote about recursive self-similarity, although his insights were somewhat rudimentary and confined to a straight line (LeClerc 103). In 1872, Karl Weierstrass examined a function that he described as “being everywhere continuous but nowhere differentiable” (6). Thirty-two years later, Helge von Koch expanded upon Weierstrass’ work by providing a geometrical definition of a similar function (Trochet). The function led to what is now called the Koch curve and the Koch snowflake. Waclaw Sierpinski expanded upon these structures through the Sierpinski triangle and the Sierpinski carpet (Grobstein). In 1938, Paul Pierre Levy described his own fractal curve, now referred to as the Levy C curve (“Levy C Curve”). Georg Cantor similarly described subsets of a line that had unusual, fractal-like properties (“History of Mathematics”). These ideas were explored and expanded upon by such researchers in the 19th and 20th centuries as Henri Poincare, Felix Klein, Pierre Fatou, and Gaston Julia (Turner). However, it was not until the advent of the computer that mathematicians became fascinated by the beauty of the fractals they were describing.

Perhaps the name most often associated with fractals is Benoit Mandelbrot. In 1975, Mandelbrot coined the term "fractal" from the Latin word *fractus*, meaning "broken" or "fractured" (Mandelbrot, *The Fractal* 81). Not only did Mandelbrot formally define a fractal, but he also used computer constructions to show the world how beautiful fractals could be. Mandelbrot, who died last year (10/14/2010) at the age of 85 years ("Benoit Mandelbrot"), applied mathematics to the complexity of the natural world. Mandelbrot argued that seemingly random mathematical shapes and forms in nature actually follow a pattern if they are broken down into a single, repeating shape (*The Fractal* 89). Mandelbrot applied his understanding of fractals to cauliflowers, coastlines, and other natural phenomena in a manner that engaged the general public.

Fractals can be classified by the degree of self-similarity they exhibit, as shown in Figures 1-3. When a fractal exhibits exact self-similarity, it is identical at different scales. These fractals are often generated by iterated functions systems that use a fixed geometric replacement rule ("Fractal," *New World Encyclopedia*). Examples of fractals that exhibit exact self-similarity are the Sierpinski triangle (see fig. 1) and the Koch snowflake. Other fractals exhibit quasi-self-similarity. In this classification, the fractal appears similar, but not identical, at different scales. At different scales, the fractal contains copies of the whole but in distorted and degenerate forms. Quasi-self-similar fractals utilize a formula or recurrence relation at each point in space. The Mandelbrot set (see fig. 2) is an example of a fractal that exhibits quasi-self-similarity (Briggs 45). The weakest type of self-similarity is the statistical self-similarity. In this type of fractal, the fractal has numerical or statistical measures that are preserved across scales. An example of a statistically similar fractal is a coastline (see fig. 3), such as that of Great Britain (Mandelbrot, "How Long" 636).

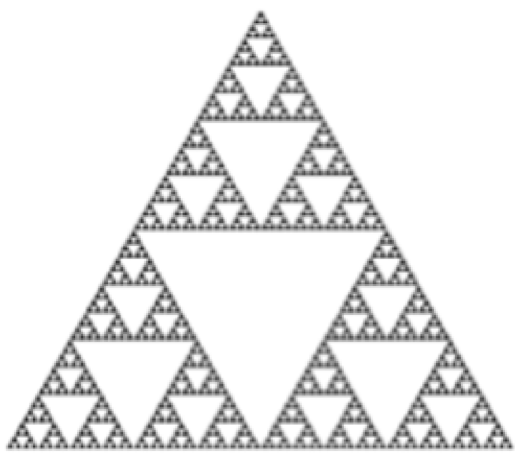

Fig. 1. Exact self-similarity, the Sierpinski triangle, from "Sierpinski Triangle" (Wikipedia; 25 May 2010; Web; 14 Nov. 2010).

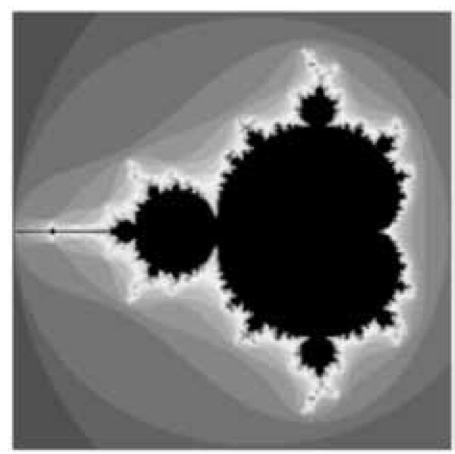

Fig. 2. Quasi-self-similarity, the Mandelbrot Set, from "Unveiling the Mandelbrot Set" by Robert L. Devaney (*Plus Magazine*; 1 Sept. 2010; Web; 14 Nov. 2010).

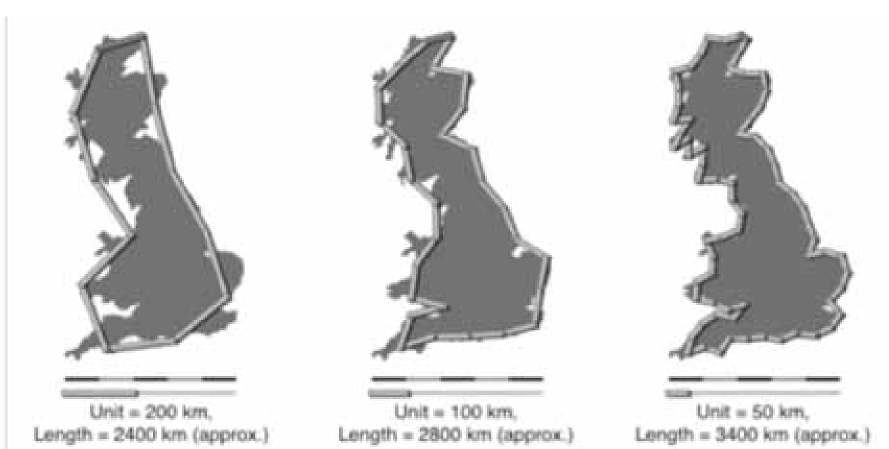


Fig. 3. Statistical similarity, the coastline of Great Britain, from "How Long is the Coastline of Great Britain?" by Benoit B. Mandelbrot (*Science*, 156; 1967; print; 637).

One of the simplest fractals, the Koch curve, provides one of the most concrete and accessible entrances into the world of fractals. Fig. 4 shows how a simple curve can be transformed into a fractal.

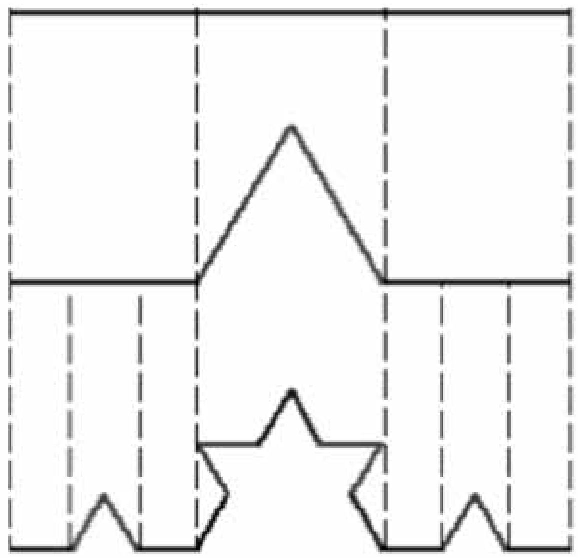

Fig. 4. The Koch Curve, from "Koch Curve" by Larry Riddle (*Classic Iterated Function Systems*; Agnes Scott College; 7 June 2010; Web; 14 Nov. 2010).

Imagine a line split into three congruent portions. Duplicate the center segment and place a copy of it over the original middle segment. Swing each of these identical twin middle segments upward on hinges connected to the opposite outer segments until their unattached ends meet to form two sides of an equilateral triangle. The result is a figure with four segments. Now repeat the process on each of the four segments, again bumping out the center third of each to form two sides of an equilateral triangle that has new sides of one-third of the segment after the first iteration. Repeat this iteration over and over. After each iteration, the length of each new segment of the figure has been reduced to one-third of the length of the previous iteration, the number of segments has increased by a factor of four, and the length of the figure has increased by a factor of 4/3.

An only slightly more complicated figure, the Koch snowflake, can be created by starting with an equilateral triangle and repeating the process described for the Koch curve on all three sides of the triangle simultaneously. The Koch snowflake is a fractal curve in which each edge of an equilateral triangle is repeatedly replaced by four congruent edges pointed outward (McGuire 14). After several iterations, objects like those shown below (fig. 5) are generated. Notice the appearance of intricate repeated visual patterns in the snowflakes at ever smaller scales. Notice also that, even though the number of segments comprising the snowflake grows, the snowflake remains in a bounded area. However, consider its perimeter, which increases after every iteration. Amazingly, if the iterative process is continued ad infinitum, the area of the snowflake will remain finite, but its perimeter will be infinite because of the concavity of its sides. Also, its boundary, though continuous, will be non-differentiable at every point of its infinite length. Thus the boundary has a quality of roughness.

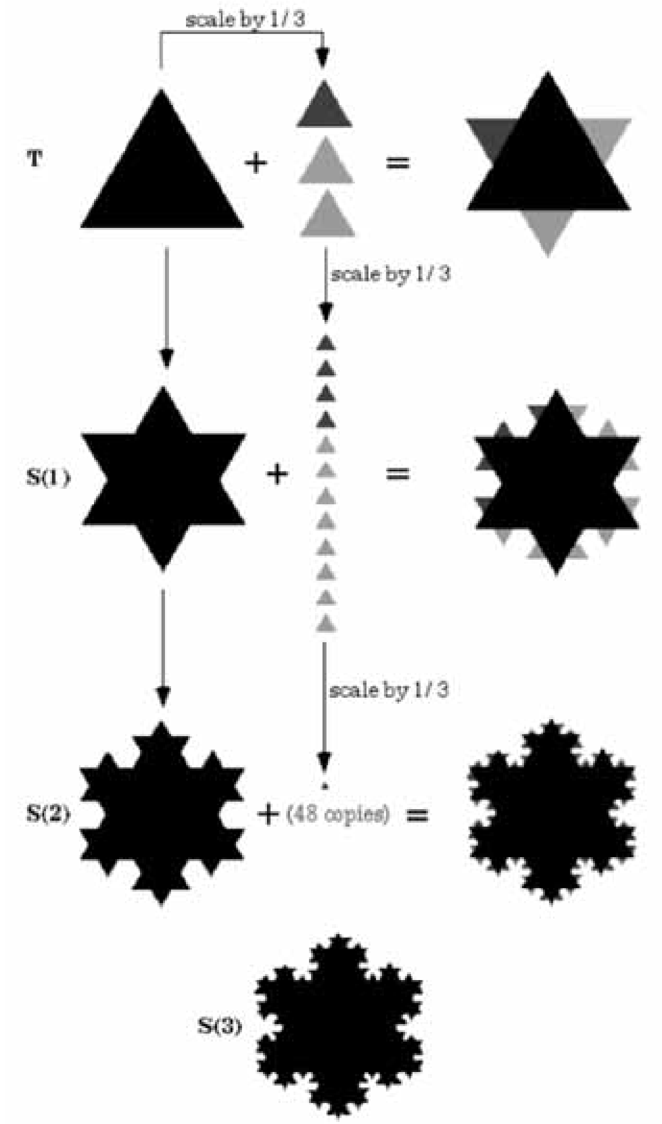


Fig. 5. Construction of the Koch Snowflake, from "Koch Snowflake" by Larry Riddle (*Classic Iterated Function Systems*; Agnes Scott College; 25 Jan. 2010; Web; 14 Nov. 2010).

**Dimensionality: The Space of Fractals**

Dimensionality refers to the complexity of a shape or the amount of space something takes up. For most shapes, dimensionality is well defined and obvious. For example, a square is two-dimensional, while a cube is three-dimensional. For the more complex geometrical structure of fractals, however, different methods of calculating dimension can yield different results (Sutherland). Three such methods are the topological dimension, the Hausdorff dimension, and the similarity dimension. The most intuitive measure of dimension is the topological dimension. However, the topological dimension does not give meaningful results for the dimensions of fractals, so Hausdorff invented another dimension, called the fractal dimension or Hausdorff dimension. Due to difficulty calculating the Hausdorff dimension, yet another dimension, the similarity dimension, was invented for ease of calculation. The similarity dimension is equal to the Hausdorff dimension for simple fractals (Weisstein).

A topological dimension is simply the standard way people have been trained to think about dimensions. From youth, people learn that lines and curves are one-dimensional, planes and surfaces are two-dimensional, and solids such as cubes are three-dimensional. Mathematically, we identify a set as *n*-dimensional when *n* independent variables are required to describe it. The dimension of the union of finitely many sets is the greatest of any one of them. The Dutch mathematician L. Brouwer defined the topological dimension of a cube as three, because any way the cube is decomposed into smaller bricks, there will always be points that belong to at least four (3 +1) bricks (qtd. in Pearse). The topological dimension is always an integer.

In contrast, the fractal dimension can be a fraction. The fractal dimension, often labeled $D$, indicates how completely a fractal fills space upon greater levels of magnification. For the Koch snowflake mentioned above, the length of the curve between any two points is infinite. No small piece of the snowflake is a line nor does it resemble a plane. In some ways, it is too large to be considered a one-dimensional object and too thin to be a two-dimensional object, suggesting that its dimensionality might lie somewhere between one and two (Lanius). This is the fractal dimension. There are two main approaches used to generate a fractal structure (Zhang  and Wang). The task can be approached from the top-down or bottom-up. In other words, the structure can be grown from a unit object or a structure can be divided into a new form.

The Hausdorff dimension, or fractal dimension, was introduced in 1918 to measure the fractional dimensions of fractal sets (Beardon 722). Typically, people think about whole-number topological dimensions of smooth objects, such as a plane or a straight line. For more complicated sets and curves, the Hausdorff dimension provides a different way to define dimension. For example, if a curve twists around and partially fills a plane, its Hausdorff dimension increases beyond one and its value approaches two the more the line fills the plane (Beardon 723). The Hausdorff dimension is calculated by examining the rate at which the surface of the object grows as the scale shrinks (Weisstein). Using this information, a fractal has been defined as a set for which the Hausdorff dimension exceeds the topological dimension (Mandelbrot, *The Fractal* 109).

The similarity dimension applies to shapes that are composed of copies of themselves whose linear size is smaller than that of the parent shape (Weisstein). The similarity dimension counts the number of copies that can fill up the shape and the size of those copies relative to the parent shape. The similarity dimension is calculated by taking the log of the ratio of the size of the original figure compared to the size of the copies and dividing by the log of the number of copies needed to fill up the space of the original figure (McGuire 19-20). It is equal to the fractal dimension for simple fractals and is generally easier to calculate than the fractal dimension.

Let's first examine how fractals challenge our common notion of dimensions of geometric objects by examining their border length. In a 1967 article in *Science*, Benoit Mandelbrot defined the dimension D of a border as

$$L(a) = ka^{1-D}$$

when $a \to 0$, where $L(a)$ is the measured length of a border using a ruler of length $a$ and $k$ is the length when $a = 1$ (Mandelbrot, "How Long" 636). The dimension D is simply $1 - m$, where $m$ is the slope of a log-log plot of the measured length of the border as a function of the length of the ruler as the ruler length approaches zero. Since the slope of this line for a Koch snowflake is constant for all ruler lengths, D can be found by simple rearrangement of equation 1. The dimension D for a snowflake from a equilateral triangle with sides of length 1, $k = 1$, can be calculated as follows when $a = 1/3$, $L(a) = 4$:

$$D = \frac{\log(a) - \log(L(a)) + log\ k}{\log(a)}$$

$$D = \frac{\log(1/3) - \log(4) + \log(3)}{\log(1/3)} = \frac{\log(4)}{\log(3)} \approx 1.26$$

This result is surprising, since common geometric figures with finite perimeters, like circles and polygons, and even complex curves with finite lengths, have dimensions of 1.

Even more surprisingly, not only have abstract hypothetical objects like Koch snowflakes been found to have fractional dimensions, but so have real object like coastlines. Lewis Fry Richardson, while studying the relationship between the length of a country's coastline and the probability of the country being involved in war, discovered that reported lengths of coastlines varied and that the measured length depends upon the scale of the measurement (qtd. in Mandelbrot, *The Fractal* 29, 33). As shown below (see fig. 6), his log-log plots of the coastline length as a function of ruler length revealed dimensions of greater than 1 (Mandelbrot, "How Long" 636).

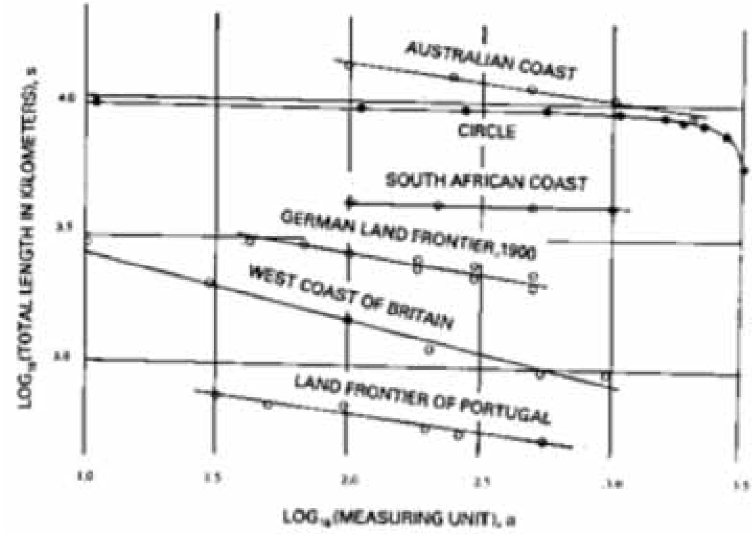


Fig. 6. Measured coastline length versus ruler length, from *Fractals: Endlessly Repeated Geometric Figures* by Hans Lauwerier, trans. Sophia Gill-Hoffsadt (Princeton: Princeton University Press; 1991; print; 30).

In 1967, Mandelbrot wrote, “I interpret Richardson’s relation as contrary to the belief that curves of dimension greater than one are an invention of mathematics” (Mandelbrot, “How Long” 637).

An equivalent similarity definition of dimension D is the number of self-similar structures $N$ with linear dimension $a$, to create the whole structure in the limit as $a$ approaches 0. This definition results in the following equation (Lauwerier 34):

$$D = \lim_{a \to 0} \frac{\log{(N(a))}}{\log{(1/a)}}$$

Additional surprising dimensions are obtained when applying this equation to the set of Cantor Dust. This set is obtained by dividing a segment into three sections, removing the middle segment, and repeating this process on the resulting set ad infinitum. For this set, the number of self-similar structures is 2 when the length a = 1/3. So, the D = log(2)/log(3) = 0.63 (Hastings and Sugihara 10). In addition, as McGuire explains, a perfectly flat sheet of aluminum foil has a dimension of 2. Crumpling the foil and then un-crumpling it leads to a statistical self-similar structure that looks like a mountain range. This structure has a dimension between 2.0 and 3.0 (McGuire 29).

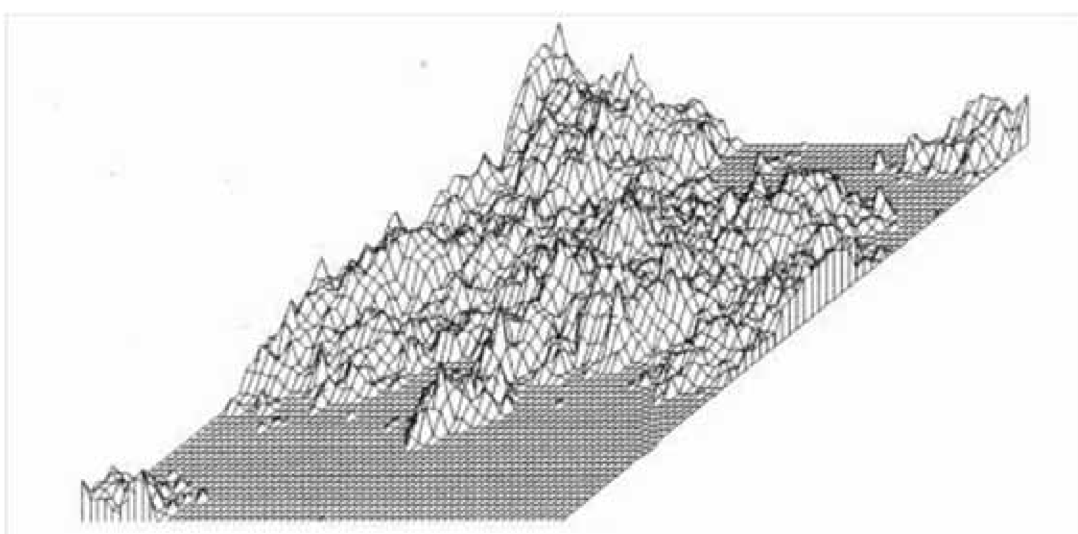

Fig. 7. Fractal mountains with fractal dimension 2.5, from *An Eye for Fractals* by Michael McGuire (Redwood City, CA: Addison-Wesley Publishing Company; 1991; print; 29).

So the set of Cantor Dust, a set of points, where any point is traditionally considered to have a dimension of 0, has a dimension between 0 and 1; the boundary of a Koch snowflake formed from line segments of a dimension of 1 has a dimension between 1 and 2; and a crumpled sheet of aluminum foil formed from a flat plane with a dimension of 2 has a dimension between 2 and 3. So these self-similar fractal objects extend our common notion of integer dimensions to fractional values.

## Fractals and Chaos Theory

Just as with fractals, the mathematics of chaos theory involves iteration of simple mathematical formulas. Although chaos theory has earlier roots, it became popularized with the invention of the computer, which made these calculations practical; fractals offered a way to visualize the results. Lorenz, in attempting to predict the weather, found that small changes in initial conditions due to rounding calculations led to large changes in the long-term outcome (qtd. in Kazantsev).

Mandelbrot found recurring patterns in data on cotton processing that eventually led to his insights about fractals (*The Fractal* 120). He concluded that the noise was patterned on a Cantor set in which the proportion of noise-containing periods to error-free periods was constant on any scale (*The Fractal* 123). Mandelbrot then applied this reasoning to the analysis of a coastline, in which the total length depended upon the scale of the measuring device ("How Long" 636-7). The coastline appears infinite in length for infinitesimally small measuring devices, for example. Mandelbrot further argued that a ball of twine appears to be a point (0-dimensional) when viewed at a great distance, a ball (3-dimensional) when viewed nearby, and a curve (1-dimensional) when viewed extremely close. These observations indicated for Mandelbrot that the dimensions of an object are relative to the observer and may be fractional. An object whose irregularity is constant across scales is considered to be a fractal. Mandelbrot published his findings in 1983 in *The Fractal Geometry of Nature,* a book that became a classic in the field of chaos theory ("History of Mathematics").

Chaos is defined as obtaining vastly different results when beginning with only slightly different starting points (Bradley). The iterative equation $z_n = z_{n-1}^2 + c$ is chaotic in that slight changes in $z_n$ or $c$ lead to vastly different results as $n$ increases. The Mandelbrot set is the set of points, $c$, in the complex plane where the above function is bounded if $z_0 = 0$. The boundary of the Mandelbrot set is a fractal with Hausdorff dimension 2 (Shishikura). Other examples of chaotic fractals are Julia sets which are sets of points $z_n$ in the complex plane such that the function stated above is bounded for a given $c$. The boundary in some Julia sets are fractals with varying Hausdorff dimensions (Weisstein).

Just as self-similarity surprisingly alters our view of dimension, a simple mathematic recursive rule, like that used to generate the Mandelbrot set, surprisingly leads to complexity. Members of the Mandelbrot set are those values of c that do not produce values attracted to infinity (McGuire 86). Once z reaches a distance from the origin of greater than 2, it is attracted to infinity. The colors in the Mandelbrot Set (see Appendix, fig. 8) are related to the number of iterations before the c value produces a z at a distance greater than 2 from the origin. Those points closer to the red side of the color spectrum reach the critical distance of 2 most quickly, while those towards the blue end of the spectrum reach this critical distance most slowly. The black points never reach the critical distance of 2 and are therefore the points in the Mandelbrot set (McGuire 86).

The surprising result is that the complexity of this set arises from such a simple recursive rule. The close-up of a region of a bulb attached to the main cardioid of the Mandelbrot set reveals just a small sample of the remarkable complexity of its boundary. Amazingly, there are an infinite number of self-similar replicates of the main structure in this finite area of 2.5 units by 2.5 units, most of which become apparent only when small areas are magnified many times!

This changes our very view of complexity, as we tend to think that complexity does not arise from simplicity. Yet the Mandelbrot set reveals that simple rules with feedback can create complex systems ("The Secret Life of Chaos"). So surprisingly, as McGuire explains, complexity arises neither from deterministic order, nor from haphazard randomness, but rather from deterministic chaos where complexity and simplicity are complementary (123). Thus complex systems can arise from simple mathematical rules that contain feedback.

## Real-World Applications

Although fractals can be appreciated simply for the beautiful pictures they create, their properties have numerous applications for mathematicians. Fractals differ from the simple lines and curves derived from many equations. They are complex patterns that are unpredictable unless the patterns are applied recursively. Mathematicians use fractals to predict complex and seemingly "random" things. Fractals approximately describe many real-world objects, like clouds, mountains, air turbulence, roots and branches of trees, as well as the veins and lungs of animals.

Scientists and engineers use fractals in their work in a variety of ways. Fractals are useful in approximating the structure of a real or imagined object. For example, a biomedical engineer might need to calculate the amount of surface area of the bronchial tubes within a human lung, or an environmentalist might use fractals to estimate the number of miles of coastline affected by an oil spill. Physicists are interested in fractals in an effort to describe the chaotic behavior of real-world phenomena such as planetary motion, the flow of liquids, the diffusion of drugs, and the vibration of airplane wings. Seismologists have begun using fractals to understand and predict earthquakes ("Fractal Geometry").

The field of nonlinear dynamics also uses fractals. In this field, the behavior of a system is described by a geometrical object in "phase space" (Gröller). This geometrical object can assume different forms, such as loops, spirals, or points, depending upon whether or not there are changes in behavior. These shapes provide meaningful data to researchers and practitioners. When the behavior of the objects is complex and not easily defined by a more basic geometrical form, fractals are useful.

The geometry of fractals has direct implications in the field of computer science. Video game designers incorporate fractals into their computer code in order to simulate randomness and more natural movements. Just as fractals show self-replications on different scales, cellular automata are computers with fractal-like properties that can potentially build copies of themselves.

Some natural forms lend themselves easily to explanation through fractals. Many natural objects display self-similar structure over an extended, but finite, scale range. Some examples include clouds, snowflakes, crystals, mountain ranges, lightning, river networks, cauliflower, blood vessels, coastlines, and many other natural phenomena (see figs. 9 and 10). Since trees and ferns are fractal in nature, they can be modeled on a computer using a recursive algorithm (Hastings and Sugihara 3). In these examples, a branch from a tree or a frond from a fern looks like a miniature replica of the whole. The connection between fractals and leaves has helped some researchers calculate the amount of carbon contained in trees ("Fractals--Hunting the Hidden Dimension"). New uses of fractals appear regularly, making fractal geometry a burgeoning and exciting field. Fractals are everywhere, and understanding them leads mathematicians in new directions that previously have been ignored.

## Conclusion

The concepts from fractals – simple recursive self-similarity generating complex structures with fractional dimensions – surround us. Self-similarity on ever decreasing scales, for example, is apparent even when examining the branching structure of tree limbs. Each subsection is a smaller replicate of the whole. Interestingly, when gazing from a distance at the thin twigs of barren tree limbs at twilight, one can almost see the thin black lines partially fill the cross sectional area, giving a hint that the dimensionality of the branches is between 1, that of a line, and 2, that of a closed figure. Likewise the fractal self-similar structures of the fine lines of blood vessels and bronchial tubes seem to fill space, contradicting our common notions of dimensions of lines.

The complexity of phenomena as dissimilar as trees limbs, clouds, and financial markets mysteriously seems to arise not from total chaos and not from a deterministic, clockwork process, but rather from a rule-based replication process of self-similar structures influenced by feedback from the environment. It seems appropriate that fractals are fascinating. They seem to reveal a link of similarity between evolution, the abstract world of mathematics, and the *real* physical and human worlds that uncover a complexity arising from simplicity. In doing so, fractals challenge our perspectives, alter our paradigms, and prompt us to pause and wonder, which, perhaps, is their greatest gift.

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## Appendix

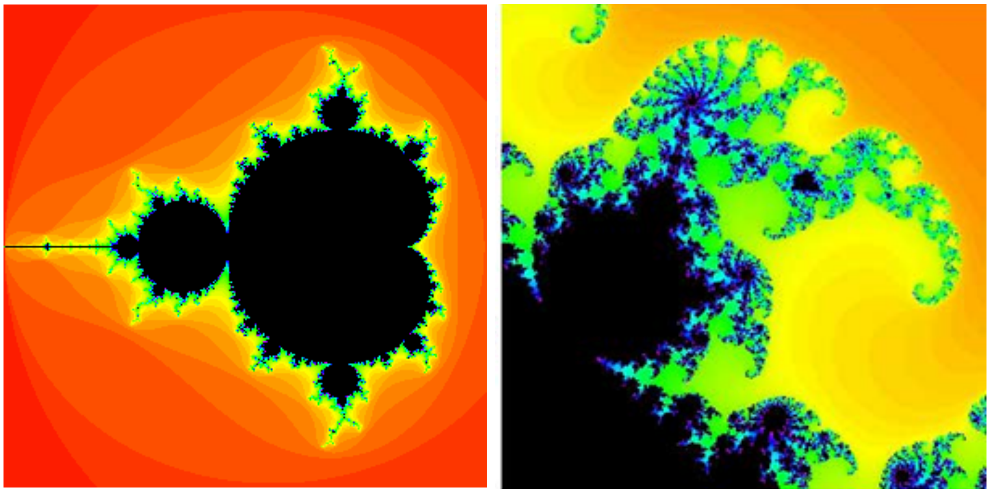

Fig. 8. The Mandelbrot Set with close-up of bulb attached to the main cardioids, from "Unveiling the Mandelbrot Set" by Robert L. Devaney (*Plus Magazine*; 1 Sept. 2006; Web; 14 Nov. 2010).

Fig. 9. Examples of fractals: simulated coral and fern leaf, from "An Introduction to the Fascinating Patterns of Visual Math: Naturally Occurring Fractals" (*Miqel.com*; Jan. 2010; Web; 10 October 2010).

Fig. 10. Peacock fan, from "11 Fascinating Fractals in Nature" (*oddee.com;* 30 Dec. 2008; Web; 14 Nov. 2010).